Momentum Space vs Position Space

Note Title 10/11/2009

position space waveftn $= \Psi(x,t)$
Momentum space waveftn $= \Phi(p,t)$

$$\Psi(x,t) = \int \Phi(p,t) f_p(x)\, dp$$

$$\Phi(p,t) = \langle f_p | \Psi(x,t) \rangle$$
$$= \int f_p^*(x)\, \Psi(x,t)\, dx$$

With $f_p(x) \equiv \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{ipx}{\hbar}\right)$,

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Phi(p,t)\, e^{\frac{ipx}{\hbar}}\, dp$$

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Psi(x,t)\, e^{-\frac{ipx}{\hbar}}\, dx$$

* Probability of finding the particle in $a < x < b$, is

$$\int_a^b |\Psi(x,t)|^2 dx$$

Probability of finding the particle's momentum between p_1 and p_2 is

$$\int_{p_1}^{p_2} |\Phi(p,t)|^2 dp$$

$\boxed{\text{Prob. 3.12}}$ Do it as a homework.

In "x" space vs In "p" space

$\hat{x}$: $x \Longleftrightarrow -\frac{\hbar}{i}\frac{\partial}{\partial p}$

$\hat{p}$: $\frac{\hbar}{i}\frac{\partial}{\partial x} \Longleftrightarrow p$

$\hat{Q}(\hat{x},\hat{p})$: $\hat{Q}(x, \frac{\hbar}{i}\frac{\partial}{\partial x}) \Longleftrightarrow \hat{Q}(-\frac{\hbar}{i}\frac{\partial}{\partial p}, p)$

$\boxed{\text{Ex.}}$ A momentum space wave fcn is given by

$\Phi(p,0) = A e^{-\frac{p^2}{2a^2}}$, at $t=0$.

What are $\langle p^2 \rangle$ and $\Psi(x,0)$?

$$1 = \int_{-\infty}^{\infty} |\Phi(p,0)|^2 dp = |A|^2 \int_{-\infty}^{\infty} e^{-p^2/a^2} dp$$

$$= 2|A|^2 \sqrt{\pi}\,\frac{a}{2} = |A|^2 \sqrt{\pi}\cdot a$$

$$\Rightarrow A = \left(\frac{1}{a\sqrt{\pi}}\right)^{\frac{1}{2}}$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} |\Phi(p,0)|^2 p^2 dp$$

$$= |A|^2 \int_{-\infty}^{\infty} e^{-\frac{p^2}{a^2}} \cdot p^2 dp$$

$$= 2|A|^2 \cdot \sqrt{\pi}\,\frac{a^3}{4} = |A|^2 \frac{a^3\sqrt{\pi}}{2}$$

$$= \frac{1}{a\sqrt{\pi}} \cdot \frac{a^3\sqrt{\pi}}{2} = \frac{a^2}{2}$$

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \Phi(p,0)\, e^{i\frac{px}{\hbar}}\, dp$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-\frac{p^2}{2a^2}}\, e^{i\frac{px}{\hbar}}\, dp$$

If we are to evaluate $\langle p^2 \rangle$ in the position space, it will be very complicated in this case.

Generalized Uncertainty Principle

For any two observables $\hat{A}$ and $\hat{B}$, that is both $\hat{A}$ and $\hat{B}$ are hermitian,

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

See the Griffiths for the proof.

For $\hat{x}$ and $\hat{p} = \frac{\hbar}{i}\frac{d}{dx}$, $[\hat{x}, \hat{p}] = i\hbar$.

Thus $\sigma_x^2 \sigma_p^2 \geq \left(\frac{\hbar}{2} \right)^2$

$$\Rightarrow \sigma_x \sigma_p \geq \frac{\hbar}{2}$$

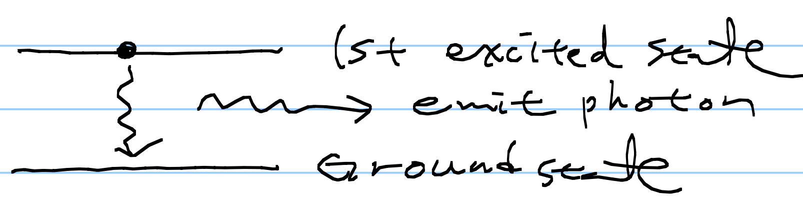
* In general, any non-commuting operators cannot have a complete set of common eigenfunctions.

⇒ Example: In Hydrogen atom to be covered in chapter 4, $\hat{H}$, $\hat{L}^2$, and $\hat{L}_z$ commute with each other, but $\hat{p}$ do not commute with these.

* <u>Energy – Time Uncertainty Principle</u>

$$\Delta t \Delta E \geq \frac{\hbar}{2}$$

A good example of this E – t uncertainty relationship is the finite line-width in optical transitions



The faster the transition time is the broader the linewidth, that is, ill-defined energy value.

* How do we derive this?

$$i\hbar \frac{\partial}{\partial t}\Psi = H\Psi$$

$$\Rightarrow \quad H = i\hbar\frac{\partial}{\partial t}$$

$$[H, t]f = [i\hbar\tfrac{\partial}{\partial t}, t]f = i\hbar[\tfrac{\partial}{\partial t}, t]f$$

$$= i\hbar\left[\frac{\partial}{\partial t}(tf) - t\frac{\partial}{\partial t}f\right]$$

$$= i\hbar f \qquad \therefore \quad [H, t] = i\hbar$$

Then $\sigma_A^2\sigma_B^2 \geq \left(\frac{1}{2i}\langle[\hat{A}, \hat{B}]\rangle\right)^2$

leads to

$$\sigma_E^2\sigma_t^2 \geq \left(\frac{1}{2i}\cdot i\hbar\right)^2$$

$$\Rightarrow \sigma_E \cdot \sigma_t \geq \frac{\hbar}{2} \Rightarrow \Delta E \cdot \Delta t \geq \frac{\hbar}{2}$$

- Here, however, $\sigma_t \equiv \sqrt{\langle t^2\rangle - \langle t\rangle^2}$, does not have any physical meaning because "t" is not an operator in Schrödinger equation

In order to get some physical meaning out of it, we need a different approach

* How does the expectation value of an observable $\hat{Q}(x, p, t)$ change over time?

$$\underline{\frac{d\langle \hat{Q} \rangle}{dt}} = \frac{d}{dt}\langle \Psi | \hat{Q} \Psi \rangle = \langle \frac{\partial \Psi}{\partial t} | \hat{Q} \Psi \rangle + \langle \Psi | \frac{\partial \hat{Q}}{\partial t} \Psi \rangle + \langle \Psi | \hat{Q} \frac{\partial \Psi}{\partial t} \rangle$$

$$= \langle \frac{1}{i\hbar} \hat{H} \Psi | \hat{Q} \Psi \rangle + \langle \Psi | \hat{Q} \left(\frac{1}{i\hbar}\right) \hat{H} \Psi \rangle + \langle \frac{\partial \hat{Q}}{\partial t} \rangle$$

$$\left\lceil i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi \right\rfloor$$

$$= \frac{i}{\hbar}\left[\underbrace{\langle \hat{H} \Psi | \hat{Q} \Psi \rangle}_{= \langle \Psi | \hat{H} \hat{Q} \Psi \rangle} - \langle \Psi | \hat{Q} \hat{H} \Psi \rangle \right] + \langle \frac{\partial \hat{Q}}{\partial t} \rangle$$

$$= \underbrace{\frac{i}{\hbar} \langle [\hat{H}, \hat{Q}] \rangle + \langle \frac{\partial \hat{Q}}{\partial t} \rangle}$$

So if $\hat{Q}$ commutes with $\hat{H}$ and does not have an explicit time dependence,

$$\frac{d\langle \hat{Q} \rangle}{dt} = 0 \quad \text{for any state.}$$

A good example: if $\hat{Q} = \hat{H}$

$$\frac{d\langle \hat{Q} \rangle}{dt} = \frac{d\langle \hat{H} \rangle}{dt} = 0$$

, which is consistent with what we have found earlier.

* If $\hat{Q}(x,p,t) = \hat{Q}(x,p)$,

$$\frac{d\langle\hat{Q}\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H},\hat{Q}]\rangle.$$

Then

$$\sigma_H^2\sigma_Q^2 \geq \left(\frac{1}{2i}\langle[\hat{H},\hat{Q}]\rangle\right)^2$$

$$= \left(\frac{\hbar}{2}\frac{d\langle\hat{Q}\rangle}{dt}\right)^2$$

$$\Rightarrow \sigma_H\sigma_Q \geq \frac{\hbar}{2}\left|\frac{d\langle\hat{Q}\rangle}{dt}\right|$$

with $\Delta E \equiv \sigma_H$, $\Delta t \equiv \dfrac{\sigma_Q}{\left|\frac{d\langle Q\rangle}{dt}\right|}$

$$\Rightarrow \Delta E \cdot \Delta t \geq \frac{\hbar}{2}$$

Here "Δt" implies the amount of time it takes the expectation value of Q to change by one standard deviation

$\Rightarrow$ If any single observable changes rapidly (Δt small), then the uncertainty in energy is always large.

$\boxed{\text{Ex.}}$ In high energy physics, Δ particle has the life time of 10^{-23} sec and the average mass of $1232\ \text{MeV}/c^2$. What is the minimum standard deviation of the measured mass spectrum?

$$\Delta E\,\Delta t \geq \frac{\hbar}{2} = 3\times10^{-22}\ \text{MeV sec}$$

$$\Rightarrow \Delta E \geq \frac{3\times10^{-22}\ \text{MeV}\cdot\text{sec}}{10^{-23}\ \text{sec}}$$

$$= 30\ \text{MeV}$$